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A VARIANT OF KANNAN FIXED POINT THEOREM IN COMPLETE CONE METRIC SPACES

Manoranjan Singha
Department of Mathematics, University of North Bengal, India

ABSTRACT

In this article a generalization of Kannan fixed point theorem has been translated in the language of complete cone metric spaces by the help of a subsequentially convergent mapping.

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Keywords: *Complete cone metric space, Sequentially and subsequentially convergent mappings, Fixed point.*

I. INTRODUCTION

It is well known that the largest (in the sense of inclusion) possible range of a real valued metric is the closed right ray $[0,\infty)$ in $\mathbb{R}$. Now $\mathbb{R}$ is a Banach space and $[0,\infty)$ is a subset of $\mathbb{R}$ having the specialties

(i) it is closed, nonempty and $[0,\infty)\neq\{\theta\}$;

(ii) $a,b\in\mathbb{R}, a,b\geq 0, x,y\in[0,\infty)\Rightarrow ax+by\in[0,\infty)$; and

(iii) $x,-x\in[0,\infty)\Rightarrow x=0$. Considering these facts into account the concept of cone metric has been developed. During this development, $\mathbb{R}$ is replaced by any Banach space and $[0,\infty)$ is replaced by any subset of the underlying Banach space bearing the properties like (i), (ii) and (iii). Huang and Zhang [1] introduced this concept and some fixed point theorems for contractive mappings were proved in this context. The results in [1] were generalized by Sh. Rezapour and R. Hamlbarani in [2]. Then a number of researchers, namely, D. Ilic' and V. Rakocevic' [3, 4], I. Beg and M. Abbas [5], Y. Song and S. Xu [6, 7, 8], I. Altun, M. Abbas and H. Simsek [9], S. Radenovic' and Z. Kadelburg [10], Ya. I. Alber and S. Guerre-Delabriere [11], N. Shahzad [12], N. Hussain and G. Jungck [13], Qingnian Zhanga and Yisheng Songb [14], M. Abbas and G. Jungck [15], C. Di. Bari and P. Vetro [16, 17], C. T. aage and J. N. Salunke [18], S. Sedghi and N. Shobe [19], S. Moradi and D. Alimohammadi [20] etc. studied fixed point and common fixed theories for different types of contractive mappings in cone metric spaces. The purpose of this article is to provide a common fixed point theorem for four self-mappings and a generalization of Kannan fixed point theorem in complete cone metric spaces. Let's begin with some definitions and results that will make the paper reader-friendly. Let E be a real Banach space and P be a subset of E. P is called a cone in E if

(1) P is closed, nonempty and $P\neq\{\theta\}$;

(2) $a,b\in\mathbb{R}, a,b\geq 0, x,y\in P\Rightarrow ax+by\in P$; and

(3) $x,-x\in P\Rightarrow x=\theta$, that is, $P\cap(-P)=\theta$. For a given cone P in a Banach space E define a partial ordering $\leq$ with respect to P by $x\leq y$ iff $y-x\in P$; $x<y$ implies $x\leq y$ but $x\neq y$, while $x<<y$ will stand for $y-x\in Int(p)$. If $x,y,z\in E$ so that $x\leq y<<z$ then $x<<z$. Let X be a non empty set and P be a cone in a Banach space E. A mapping $d:X\times X\rightarrow E$ is called a cone metric if

(1) $0\leq d(x,y)$ for all $x,y\in X$ and $d(x,y)=0$ iff $x=y$.

(2) $d(x,y)=d(y,x)$ for all $x,y\in X$

(3) $d(x,y) \le d(x,z) + d(z,y)$ for all $x,y,z \in X$. This mapping d is called a cone metric on X and the ordered pair (X,d) is called a cone metric space. A sequence $\{x_n\}$ in the cone metric space (X,d) is said to converges to $x \in X$ if for any $c \in E$ with $\theta << c$ there is a natural number N such that $d(x_n, x) << c$ for all $n \ge N$. A sequence $\{x_n\}$ in the cone metric space (X,d) is said to be a Cauchy sequence if for any $c \in E$ with $\theta << c$ there is a natural number N such that $d(x_n, x_m) << c$ for all $n,m \ge N$. If every Cauchy sequence in a cone metric space is convergent then it is called complete cone metric space. It is observed that $Int(P) + Int(P) \subset Int(P)$ and $\lambda Int(P) \subset Int(P)$, where P is a cone in some real Banach space and $\mathbb{R} \ni \lambda > 0$. A cone P in a Banach space E is called totally ordered if for any $x, y \in E$ either $x - y \in P$ or $y - x \in P$, that is, either $y \le x$ (in this case we write max $\{x,y\} = x$) or $x \le y$. We define a binary operation $\circ$ on a totally ordered cone P by $a \circ b = max\{a,b\}$, $\forall$ $a,b \in P$, then it can be shown that $\circ$ is associative, commutative and continuous. The binary operation $\circ$ is said to satisfy $\alpha -$ property if there is a positive real number α so that $a \circ b \le \alpha max\{a,b\}$ for all $a,b \in P$. Two mappings A and S from a cone metric space (X,d) into itself are said to be weakly compatible if they commute at their points of coincidence, that is, $Ax = Sx$ for some $x \in X$ implies that $ASx = SAx$. A mapping T from a cone metric space (X,d) into itself is called sequentially convergent if convergence of $\{Ty_n\}$ implies that of $\{y_n\}$, for any sequence $\{y_n\}$ in X; T is said to be subsequentially convergent if convergence of $\{Ty_n\}$ implies existence of a convergent subsequence of $\{y_n\}$, for any sequence $\{y_n\}$ in X.

II. A FIXED POINT THEOREM

Theorem: *Let T, S be self mappings on a complete cone metric space (X,d) of which T is continuous, one-to-one and subsequentially convergent. If*

$$d(TSx, TSy) \le \lambda[d(Tx, TSx) + d(Ty, TSy)]; x, y \in X,$$

where $\lambda \in [0, \frac{1}{2})$ then, S has a unique fixed point.

Proof. Let x_0 be an arbitrary point in X. For $n \ge 1$, define the iterative sequence $\{x_n\}$ by:

$$x_{n+1} = Sx_n, x_n = S^n x_0, \text{for} n \in \mathbb{N}$$

Then by the given hypothesis,

$$d(Tx_n, Tx_{n+1}) = d(TSx_{n-1}, TSx_n)$$
$$\le \lambda[d(Tx_{n-1}, TSx_{n-1}) + d(Tx_n, TSx_n)]$$
$$\le \lambda[d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})] \text{and so,}$$
$$d(Tx_n, Tx_{n+1}) \le \frac{\lambda}{1-\lambda} d(Tx_{n-1}, Tx_n)$$
$$= kd(Tx_{n-1}, Tx_n), \text{where} k = \frac{\lambda}{1-\lambda}$$
$$\le k^n d(Tx_0, Tx_1)$$

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Therefore for $m,n\in\mathbb{N}$, $m>n$,

$$d(Tx_m,Tx_n)\le d(Tx_m,Tx_{m-1})+d(Tx_{m-1},Tx_{m-2})+........+d(Tx_{n+1},Tx_n)$$
$$\le(k^{m-1}+k^{m-2}+.........+k^n)d(Tx_0,Tx_1)$$
$$\le\frac{k^n}{1-k}d(Tx_0,Tx_1)$$

Then for any $\theta<<c$, there is $N\in\mathbb{N}$ such that $d(Tx_m,Tx_n)<<c\ \ \forall\ \ m,n\ge N$. So, $\{Tx_n\}$ is a Cauchy sequence in the complete cone metric space (X,d) and hence convergent therein; consequently, since T is subsequentially convergent, $\{x_n\}$ has a convergent subsequence $\{x_{n(k)}\}$ converges to some point $u\in X$. Continuity of T ensures that $\lim\limits_{k\to\infty}Tx_{n(k)}=Tu$. Therefore,

$$d(TSu,Tu)\le d(TSu,TS^{n(k)}x_0)+d(TS^{n(k)}x_0,TS^{n(k)+1}x_0)+d(TS^{n(k)+1}x_0,Tu)$$
$$\le\lambda[d(Tu,TSu)+d(TS^{n(k)-1}x_0,TS^{n(k)}x_0)]$$
$$+(\frac{\lambda}{1-\lambda})^{n(k)}d(TSx_0,Tx_0)+d(Tx_{n(k)+1},Tu)$$
$$\le\lambda d(Tu,TSu)+\lambda(\frac{\lambda}{1-\lambda})^{n(k)-1}d(Tx_0,Tx_1)$$
$$+(\frac{\lambda}{1-\lambda})^{n(k)}d(Tx_1,Tx_0)+d(Tx_{n(k)+1},Tu)\text{andhence,}$$
$$d(TSu,Tu)\le(\frac{\lambda}{1-\lambda})^{n(k)}d(Tx_0,Tx_1)+\frac{1}{1-\lambda}(\frac{\lambda}{1-\lambda})^{n(k)}d(Tx_1,Tx_0)$$
$$+\frac{1}{1-\lambda}d(Tx_{n(k)+1},Tu)\to\theta\text{as}k\to\infty$$

Choose a natural number $N>1$ so that $d(TSu,Tu)<\frac{1}{N}d(TSu,Tu)$, then $(\frac{1}{N}-1)d(TSu,Tu)\in P$, where P is the underlying cone. Since $\frac{1}{N}-1<0$ therefore, $d(TSu,Tu)=\theta\Rightarrow TSu=Tu$. But T is one-to-one therefore $Su=u$.

Let $u_1\in X$ be also a fixed point of S then $Su_1=u_1$. Now,

$$d(Tu,Tu_1)=d(TSu,TSu_1)$$
$$\le\lambda[d(Tu,TSu)+d(Tu_1,TSu_1)]$$
$$=\lambda[d(Tu,Tu)+d(Tu_1,Tu_1)]$$
$$=\theta$$

$\Rightarrow Tu=Tu_1\Rightarrow u=u_1$, since T is one-to-one. So, we are done.

We conclude with the following

Corollary: *In the above* ***Theorem****, replacement of subsequentially convergent T by sequentially convergent T does not change the conclusion; it changes the way of reaching to that conclusion. Because, in this case,*

$\lim_{n\to\infty} S^n x_0 = \text{The fixed point of S, for every } x_0 \in X.$

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